Entropy geometry in Vertex Models

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Abstract

We study higher order vertex models on a bounded domain with Domain Wall -like boundary conditions. After first establishing the dynamic versions of the models it is shown that limit shapes prevail outside DWBC, embedded Ice-dynamics is not a prerequisite for this phenomenon to appear and especially in the 19-vertex context a wide variety of limit geometries can appear due to multiple interacting subactions.

1 Introduction

In the **Vertex models** of Statistical Mechanics the spin variables at vertices are replaced by arrow orientations on the edges of a given lattice. A **vertex rule** determines the allowed local vertex configurations. The best know such model is the **six-vertex** one defined on the square lattice (Ice model).

This vertex rule can be relaxed in various ways, some of them even physically meaningful. If one allows also unoriented edges but keeps the flux at each vertex zero, then perhaps the two best know variants in the physics literature are **15-vertex** and **19-vertex models.**

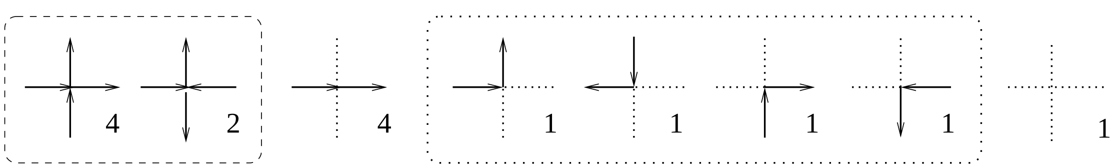

Figure 1: The 15-vertex rule: allowed vertex configurations with multiplicities. For the 19-vertex rule augment the center frame with the four missing rotations. Six-vertex rule consists of the frame on the left.

2 Set-up

The models live on a square in $\mathbf{Z}^2$, oriented with the lattice lines. This enables comparison to various earlier six-vertex results. We could impose the **Domain Wall Boundary Condition** (DWBC): all side boundary arrows pointing in, out at the top and bottom.

However the 15-vertex rule requires more: with DWBC it reduces to six-vertex rule. The 19-vertex rule is still highly nontrivial under DWBC.

The vertex rules admit **height function** which maps a configuration to a unique Lipschitz surface over it. It also enables one to convert the set of configurations sharing a boundary into a **distributed lattice**. The elementary actions allow one to convert the configurations into each other.

To be able to compute the equilibria, we dynamize the rules. Using the height formulation one can distill the minimal sets of actions for the vertex rules. Equivalently one can think of the dynamic rule as **deposition/sublimation scheme** of unit cubes and $1 \times 1 \times 2$ pieces.

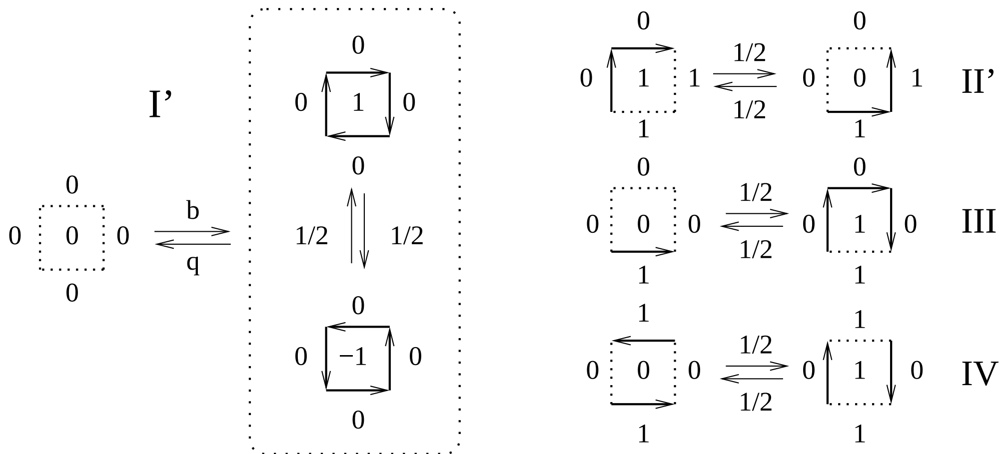

Figure 2: Minimal set of 19-vertex elementary actions (taken together with all reflections and rotations). Heights are indicated by the sides and inside of the lattice unit squares. Arrows in between indicate the allowed transitions together with their probabilities. The framed action is all that is needed for the six-vertex. This action together with II' and its rotation by π and no reflections generate the 15-vertex configurations.

These generate all the legal configurations compatible with the boundary condition from any one sample.

Theorem 1 (Irreducibility and ergodicity)**.** *Given a 19-vertex configuration on a bounded domain, any other legal configuration with the same boundary condition can be generated from the former using a finite sequence of elementary actions I'-IV. A strict subset of actions will not suffice. When $0 < b, q < 1$ the Markov Chain on the graph of legal configurations is ergodic.*

On the indicated subsets of actions the same conclusions hold for six-vertex and 15-vertex rules. A unique stationary measure exists for all.

3 Results

The simplest way to generate the DWBC-compatible set of configurations on the square is from a configuration whose height is geometrically a SW-NE oriented **ridge roof**. Minimal planing of the ridge corresponds to replacing the four SW and NE corner arrows by unoriented edges and connecting them with a neighboring pair of blank staircases. This is a non-DWBC for the 15-vertex case and strictly beyond six-vertex.

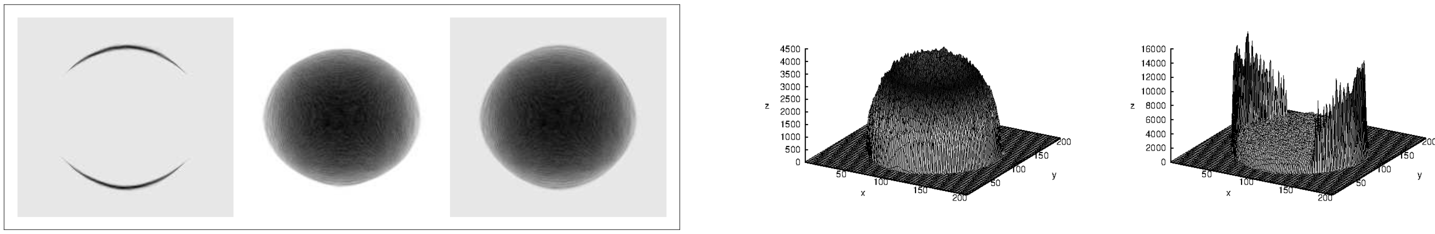

Figure 3: 15-vertex non-DWBC equilibrium. Left: densities of actions II', I' and I'+II'. The square is tilted by $\pi/4$; the ridge is horizontal. 3d plots: the same data, in the center the densities superimposed. On the right the distribution sum with type II' weighted tenfold. Iterates 61-70.000, 206^2 square.

One can perturb the ridge roof further away from DWBC by placing blank terraces (height across them vanishes) in the form of further blank zig-zags parallel to the sides. Then the boundary condition is however on the diamond outscribing the square. One can readily see more exotic equilibria (below) and unlike above it is not clear that the action II' contribution is vanishing in the scaling limit.

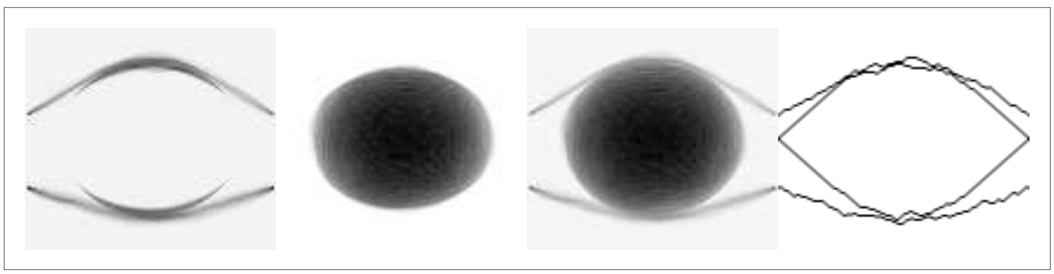

Figure 4: Equilibria under 15-vertex dynamics from terrace-type initial conditions. Actions II', I' and I'+II'. 106^2 square, 20-50.000 iterates. Right: snapshot at the termination: just unoriented edges.

The 19-vertex is by far the most complex rule here. Even with just the two parameters a variety of phenomena emerge. We present here only samples of the DWBC case.

When q vanishes 19-vertex reduces to six-vertex. There is only one active subaction of I', the reversal of unidirectional unit squares, as seen in bottom line of the Figure below. The normalized Ice weights are one i.e. we are at the 1-enumeration point of ASM's in the disordered phase and the limit shape is a bit off from a circle ($\Delta = \frac{1}{2}$, [**?**]).

In the middle row of Figure 5 illustrates the case $q = b > 0$. All points have the same equilibria but relaxation rates differ. All actions are at play but their supports are quite different. Each is rendered for maximum contrast. The relative intensities are III>II'>I'≈IV.

The top row of Figure 5 shows action equilibria for $b = 0,\ q = 1$. Under this rule any appearing unidirectional 1-cycle is immediately killed. There are no reversual of these cycles involved i.e. no Ice-dynamics takes place. For this **Anti-Ice** the largest distribution II' is also very steep and likely leading to a limit shape.

Each row below captures the distribution and strength of the randomness that emerges from the most elementary complete set of perturbations acting on the configurations. It describes the **geometry of the entropy generation** for the given boundary condition and parameter values. The configuration level limit shapes should form as unions of the action distribution supports.

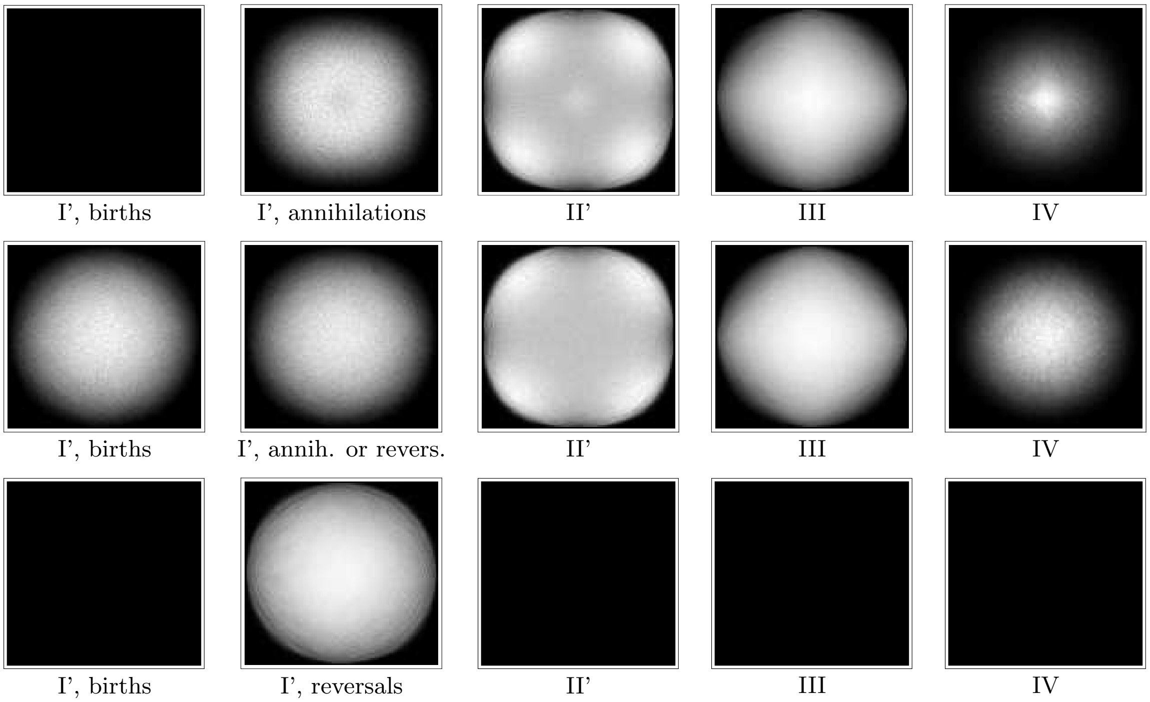

Figure 5: Equilibrium action densities for the 19-vertex rule. DWBC on 106^2 square. Lighter is more active while black means no activity. Individually scaled for contrast. Bottom row: 1-weight Ice ($(b, 0)$, any b). Middle row: $q = b > 0$. Top row: Anti-Ice ($(b, q) = (0, 1)$). Caption subactions as in Figure 2.

Various further parametrizations are available. By skewing the action III probabilities one can microscopically alter the lengths of directed paths and the local density. Together with the other actions this can lead macroscopically to rather striking differences in equilibria:

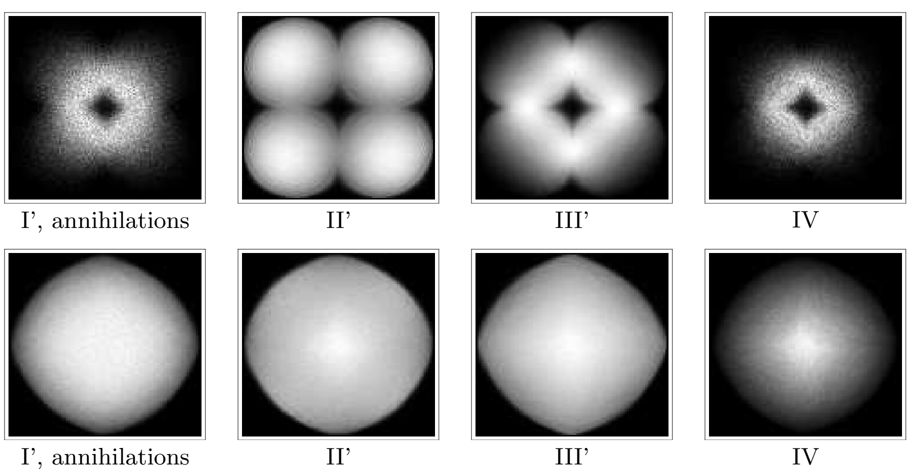

Figure 6: Densities close to $(b, q) = (0, 1)$ under skewed action III probabilities. Top row: path straightening favored 7:1. Bottom row: straightening disfavored 1:7. Between these two rows one could insert the top row of Figure 5 (without the blank square); Anti-Ice corresponds to the balanced case. Data as above.

References

[1] F. Colomo, A. Pronko. arXiv: 0907.1264v2, 2009.

[2] K. Eloranta. arXiv: 1710.03609, 2017 and 1807.07567, 2018.