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Abstract

We study subshifts of finite type on $\{0,1\}^{\mathbf{Z}^2}$ of the form where in a finite window there is always either one on no 1's present. The problem is converted to a tiling/covering problem on the $(n\mathbf{Z})^2$ lattice and its shifts. In this set-up we consider the uniqueness of the equilibrium measure, density of 1's at equilibrium and the topological entropy. Further insight is achieved by realizing that the equilibrium measure is the ground state of an extremely simple probabilistic cellular automaton. Our study indicates critical behavior in this one parameter class of rules.

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Introduction

Consider the set-up where one of the symbols 0 or 1 is assigned at every site on the two-dimensional integer lattice. Suppose furthermore that we impose a condition on what the density (frequency) of 1's can be in a window of fixed size. This a subshift of finite type, non-empty except in trivial cases yet many of its properties like the size of the subshift (topological entropy) or ergodic properties of the coordinate shifts become rather difficult questions in the general case.

This work is a continuation of the work in [E1] where the case of constant density in a rectangular window was treated. The subshifts in there have a tiling-like quality with certain weak periodicity properties which makes their rigorous analysis possible. The models exhibit extremely long range order characteristic of zero temperature systems.

In this piece we aim to advance to the positive temperature realm by allowing fluctuation in the density i.e. multiple values for the density. This leads to a rich class of systems even under the further assumption that each window has only one or no 1's. In particular the hard square gas model [B] is incorporated. We study especially the problems of characterizing the equilibrium measure and estimation of the topological entropy in the context of these systems.

We also present a dynamical approach to these models. Apart from the usual lattice shifts involved there is non-trivial but nevertheless extremely simple probabilistic cellular automaton dynamics for which the equilibrium measures of the subshift are ground states. This in turn has connection via voter models to critical behavior which we indicate at the end.

1. Subshifts and tilings

Our configuration spaces are closed subsets of the set $X = \{0,1\}^{\mathbf{Z}^2}$ equipped with the standard topology. The actions are subshifts of the full shift (X, σ_h, σ_v) where $\sigma_.$ are the usual coordinate shifts:

$$\sigma_h(x)_{(i,j)} = x_{(i+1,j)} \quad \text{and} \quad \sigma_v(x)_{(i,j)} = x_{(i,j+1)}, \qquad x \in X.$$

These actions obviously commute. To illuminate the form of restriction in our subshifts we consider a couple of "canonical" examples.

Example 1.1.: Take $\{0,1\}^{\mathbf{Z}^2}$ with the restriction that in every 1×2 and 2×1 window there must be exactly one 1. This system is of course very restricted, the set of configurations consists of exactly two points, the checkerboard and its shift. Note that they can be simply generated from a diamond-tile (i.e. square with sidelength $\sqrt{2}$ tilted 45 degrees) where the center is 1 and the vertices are the nearest neighbor 0's. Call it a **1-tile**.

Example 1.2.: Take again $\{0,1\}^{\mathbf{Z}^2}$ with the restriction that in every 2×2 window there is exactly one 1. The set of allowed configurations, which is now uncountable but still quite trivial can be simply generated using square 1-tiles of sidelength two with symbol 1 at the center. If we instead impose the density condition in a 3×3 window the same construction clearly applies but the tiles are now of sidelength three with 1 in the middle etc.
It is easy to see that the examples in 1.2. are zero topological entropy and indeed quite rigid. For details like the topological non-transitivity as well as more general zero-entropy models see [E1].

Examples 1.1 and 1.2. continued: The set of allowed configurations in Example 1.1. is exactly the one that we get by generating a tiling from the allowed 1-tile on the square lattice or its dual lattice. Suppose now that we can also use a **0-tile** which is like 1-tile but 1 at the center replaced by 0. So the allowed densities in a window are now 0 and 1/2. If we generate the configurations on the lattice and its dual using these two tiles it will be a superset of the original set, in fact an uncountable set.

Note however that the tilings generated from 0 and 1-tiles admit a further extension. If we encounter anywhere in a configuration a neighborhood where there are four 0-tiles in a 2×2 arrangement, their common corner, which is 0, could be **flipped** into a 1. Call the set of configurations generated using 0 and 1-tiles and the flipping process $X_{\{0,1\}}$. It is easy to see that this set of configurations is also characterized by the rule "in every 1×2 and 2×1 window there is at most one 1". This the classical model of hard square gas (the hard squares being our 1-tiles).

Figure 1 here

In the case of larger window like in Example 1.2. the 0-tile is defined analogously and the difference to the Example 1.1. comes from the fact that there are more than two sublattices of $\mathbf{Z}^2$ involved. In the case of a 2×2 window there are four of them on which we have to consider the possibility of a flip (as the original tiling was on $\{0,1\}^{(2\mathbf{Z})^2}$). In the case of the 3×3 window there are nine sublattices.

The construction indicated can be summarized as follows.

Proposition 1.3.: *Suppose that X_n is the subshift of the full shift X with the property that in every $n \times n$ window there is either one or no 1's, $n \geq 2$. Then the set of allowed configurations is generated from 0 and 1-tiles of sidelength n together with subsequent flipping at the allowed sites.*

2. Equilibrium and entropy

We now proceed to consider the equilibrium properties of the models. Specifically we study the simplest of the models in the previous section, the hard square gas model. The purpose of presenting the other models is that the same ideas can be immediately applied to them. Extensions needed to this are pointed along the way.

To get a basic idea about the configuration space before we dwell into the equilibrium measure(s) it is useful to parametrize it using densities of 0-tiles on even and odd lattices. We denote these by ρ_e and ρ_o. They assume values between 0 and 1. The set of configurations with well-defined densities of 0-tiles is not closed but for every ergodic measure it is of full measure. The set of such configurations can be represented as in Figure 2.

Figure 2 here

The configurations in the shaded area are forbidden as they are too dense in 1-tiles. The configurations on its boundary, the decreasing diagonal, consist of uniform domains of 1-tiles on each of the two lattices in such an arrangement that the boundary area between the domains of even and odd 1-tiles has vanishing density. This is because every pair of neighboring 1-tiles from different lattices has between them at least an area 1/2 which does not belong to any other similar pair and there is a 0-tile corresponding to it. Hence if there is a positive density of such pairs the inequality $\rho_e + \rho_o > 1$ will prevail. The corner $(1, 1)$ is trivial as it corresponds to configurations essentially empty of 1-tiles.

A central object in the study of lattice models is the **measure of maximal entropy** or the **equilibrium measure**. By this one means a measure on the configuration space which maximizes the measure-theoretic entropy thereby giving the **topological entropy**:

$$h_{top} = h_{\mu^*} = \sup_{\mu \in \mathcal{M}} h_\mu$$

Here $\mathcal{M}$ is the set of all translation (σ_h and σ_v) invariant measures on the configuration space.

We are ultimately interested in the unconditional measure of maximal entropy and the configurations generic to it but let us first see how the measure of maximal entropy behaves when conditioned to be supported by configurations with fixed density (ρ_e, ρ_o). Denote the corresponding conditional topological entropy $h(\rho_e, \rho_o)$.

The the sets of configurations at the corner $(1,1)$ and on the diagonal $\rho_o + \rho_e = 1,\ \rho_e \in [0,1]$ are of zero entropy. The former follows from the fact that positive topological entropy requires a positive density both 0 and 1-tiles and the latter from the fact that the boundary between even and odd domains of 1-tiles is of zero density. In the rest of the triangle the entropy is positive. Off these singularities the entropy behaves asymptotically as $O(\epsilon \ln \epsilon)$ where ϵ is the distance from either of the zero entropy subsets.

Along the right edge the the diagram the topological entropy is given by the entropy of Bernoulli distribution with parameter ρ_o, $-\rho_o \log \rho_o - (1-\rho_o)\log(1-\rho_o)$. The formula for the top edge is the same with parameter ρ_e.

Perturbation from these edges can be quantified as follows.

Theorem 2.1.: *For large ρ_e the conditional topological entropy $h(\rho_e, \rho_o)$ is maximized at $\rho_o = 1 - \frac{1}{2}\rho_e^4 = \frac{1}{2} + 2(1-\rho_e) + o(1-\rho_e)$.*

Proof: A site on the odd lattice is free to be flipped if it is not blocked by one or more neighboring 1-tiles on the even lattice. So the free area on the odd lattice has density $\rho_D = \Pr(\{\text{odd site free to be flipped}\}) = \rho_e^4$ where the assumption that the 0-tile distribution on the even lattice is Bernoulli(ρ_e) has been used. Still under the Bernoulli-assumption the entropy contribution from the odd lattice will be

$$h_o = -\frac{\rho_D}{2}\left(\left(\frac{1-\rho_o}{\rho_D}\right)\ln\left(\frac{1-\rho_o}{\rho_D}\right) + \left(1 - \frac{1-\rho_0}{\rho_D}\right)\ln\left(1 - \frac{1-\rho_o}{\rho_D}\right)\right)$$

where a scaling has been performed to ensure correct final 1-tile density on the odd lattice. This expression is maximized at $\rho_o = 1 - \frac{1}{2}\rho_e^4$ which expands at $\rho_e = 1$ as given. ∎

Remark: There is of course the symmetric statement for the perturbation from the top edge. It is perhaps worth pointing out that the quartic given in the Theorem intersects the diagonal approximately at $(0.7976, 0.7976)$ which as we will see is surprisingly close to the point that carries the configurations generic to the (unconditional) measure of maximal entropy. So even though the approximation is strictly valid only in the neighborhood of the boundary it is not far off in the interior.

Because of our representation of the subshifts in terms of tilings and coverings there is a special way of computing the topological entropy.

Proposition 2.2.: *The topological entropy of the hard square gas is given by*

$$h = \frac{1}{2}\left(h_{even} + \Pr(D_4)\log 2\right). \tag{2.1}$$

Here h_{even} is the entropy of the tiling on the even lattice as given by the projection of the measure of maximal entropy onto it and D_4 is a 2×2 arrangement of 0-tiles in it.

Proof: Suppose μ^* is a measure of maximal entropy and μ_e^* is its marginal on the even lattice. From the even lattice the entropy contribution is then $h_{\mu_e^*}$. The contribution from the odd lattice comes from sites where a 0-tile is not forced. These are the sites centered at D_4-diamonds. At these sites maximal entropy is attained by assigning the 1-tile from a Bernoulli(1/2)-distribution. Hence the entropy contribution from such site is $\log 2$. Finally the even and odd lattices are 1/2-thinnings of the $\mathbf{Z}^2$-lattice, hence the factor 1/2. ∎

The measure of maximal entropy doesn't need to be unique but in the case of hard square gas it is. This follows from the Dobrushin criterion ([DS], [RS]). Using this knowledge we now proceed to estimate the topological entropy and the density of 1-tiles in its generic configurations. Before going into that it should be remarked that the topological entropy has been computed to a good accuracy. Milosevic et. al. ([MSS]) give the following value

$$h_{[MSS]} = 0.40749510126068. \tag{2.2}$$

However the methods used in that and other similar numerical studies do not seem to illuminate much the generic structure of the configurations involved.

Theorem 2.3.: *The density of 1-tiles at the equilibrium (on either lattice) is in the interval* $(0.21367, 0.25806)$.

Remark: The exact values of the endpoints can be obtained from the Proof but we prefer to give the statement in this more explicit form. Note that the (crude) extrapolation 0.7976 in the Remark after Theorem 2.1. gives 1-tile density 0.2024, not very far from being in the interval!

Proof: Let ρ_e be again the density of 0-tiles on the even lattice and let c denote the expected number of 0-tiles that a 1-tile forces on the odd lattice. Since exactly half of the non-forced sites will be 1-tiles it must by the uniqueness of the measure of maximal entropy hold that $(2+c)\rho_e = 1$. Hence under it

$$\Pr(0\text{–tile}) = \frac{1+c}{2+c} \qquad \Pr(1\text{–tile}) = \frac{1}{2+c} \qquad \Pr(D_4) = \frac{2}{2+c}$$

on both lattices. D_4 is a 2×2 0-window as in the entropy formula of Proposition 2.2.

The entropy of any distribution of tiles on the even lattice with 0-tile density ρ_e is bounded from above by the entropy of the Bernoulli distribution with parameter ρ_e. Hence the total entropy at that 0-tile density level is bounded from above by

$$\frac{1}{2}\left(h_{B(\rho_e)} + \frac{2}{2+c}\log 2\right) = \frac{1}{2}\left(-\frac{1}{2+c}\log\frac{1}{2+c} - \frac{1+c}{2+c}\log\frac{1+c}{2+c} + \frac{2}{2+c}\log 2\right).$$

This expression bounded by $h_{[MSS]}$ yields an upper bound for c, 2.6801 which in turn gives the lower bound for $1-\rho_e$.

The upper bound for $1-\rho_e$ follows from a lower bound for c which we establish using a monotonicity argument. The 1-tiles on say the even lattice are Bernoulli(1/2)-distributed on the non-forced sites. Call this set F and pick a site on it which has a 1-tile. How many sites will this 1-tile block? Let F' be a superset of F. Then clearly $\mathbf{E}(c|\ F) \geq \mathbf{E}(c|\ F')$ as in a bigger domain the 1-tile is more likely to share the blocking with a nearest neighbor 1-tile. Hence a lower bound is obtained by calculating the blocking for a 1-tile in a half-space. Enumerating the 2^8 neighborhoods and weighting them uniformly according to the Bernoulli(1/2)-distribution we get the lower bound for c: 15/8. This in turn implies the given upper bound for $1-\rho_e$, 8/31. ∎

To illustrate the usefulness of the tiling representation we now briefly show some results on the numerical estimation of the topological entropy and the density of tiles. In a similar fashion other statistics of the approximate measure of maximal entropy could be extracted just as easily.

As a first approximation one could distribute the tiles on the even lattice according to Bernoulli(ρ_e)-distribution. Then $\Pr(D_4) = (1-\rho_e)^4$ and maximizing then the quantity

$$\frac{1}{2}\left(-\rho_e\log\rho_e - (1-\rho_e)\log(1-\rho_e) + (1-\rho_e)^4\log 2\right),$$

gives a lower bound

0.392421

for the topological entropy. The 1-tile density on the even lattice corresponding to this is approximately 0.170219. However the 1-tile density on the odd lattice is now different, 0.237041. If we impose a further condition, that the densities must agree, the numerical optimization yields the slightly worse lower bound 0.392125 at the common density level 0.20151.

To enable possible long term order to enter into the approximation scheme let us consider Bernoulli-distributed 2×2 tile blocks. As there are 16 such blocks and their probabilities, p_i, must add up to one, our optimization problem has 15 free variables. The quantity to be maximized is now

$$\frac{1}{2}\left(-\frac{1}{4}\sum p_i \log p_i + \Pr(D_4)\log 2\right)$$

where the factor $1/4$ stems from the blocks covering four sites.

Although we know from the uniqueness of the measure of maximal entropy that its generic configurations must be statistically isotropic i.e. rotation invariant on the lattice, it does not imply that this kind of configurations maximize the expression above. However in numerical optimization we achieved exactly the same level of entropy for the full 2×2 -model and the isotropic one. For this reason we do not give the expression for $\Pr(D_4)$ for the full model. For the isotropic model it stands as

$$\frac{1}{4}\left(p_0 + 2\left(p_0 + 2p_1 + p_{21}\right)^2 + \left(p_0 + 3p_1 + 2p_{21} + p_{22} + p_3\right)^4\right).$$

The first index of p refers to the number of 1-tiles in the 2×2-block and indices 21 and 22 refer to the bar and cross arrangements of 1-tiles in the case of two 1-tiles. The entropy achieved using optimal Bernoulli-distributed 2×2 -blocks is

$$0.39877$$

with the corresponding block distribution being approximately

$$(p_0, p_1, p_{21}, p_{22}, p_3, p_4) = (0.499, 0.069, 0.031, 0.019, 0.013, 0.009).$$

The 1-tile densities are again different, 0.199 on the even lattice and 0.225 on the odd lattice.

This procedure of going into higher order Bernoulli blocks can be continued to provide arbitrarily good lower bounds for the topological entropy (this procedure can even be automated). We did not proceed to do this but only for comparison computed the entropy bound using 3×3 -blocks and the ideas explained above. This yielded the value

$$0.40140 \tag{2.4}$$

with corresponding 1-tile densities 0.2073 and 0.2254 (note that these are approaching or are already in the density bracket indicated in Proposition 2.3). Note also that the value (2.4) is just 0.006 below the Milosevic et. al. estimate (2.2). This level of accuracy from fairly small blocks (on top of the uniqueness result quoted earlier) further confirms that the model does not have significant long term order.

We also note that the idea of Proposition 2.2. applies immediately to the more general models. E.g. for the rule "at most one 1 in a 2 × 2 window" the formula stands as

$$\frac{1}{4}\left(h_1 + \Pr\left(D_4^2\right) h_2 + \Pr\left(D_4^3\right) h_3 + \Pr\left(D_4^4\right) \ln 2\right) \quad . \tag{2.3}$$

Here one fills in the sublattices from 1 to 4. h_i are the corresponding entropies for the sublattice measures and $\Pr\left(D_4^i\right)$, which scale the contribution by the density of sites available at the i^{th} step, are evaluated after the steps upto $i-1$ have been performed. As an example of this procedure we just mention that if the fill-ins are done Bernoulli(p_i) on the available area, then a simple optimization of (2.3) yields the lower bound 0.302 for the topological entropy of the given rule (this is likely not to be very tight).

3. A probabilistic cellular automaton and criticality

The model considered so far has been static i.e. there has been no time evolution associated to it. There is though a very natural one which we now formulate.

The proof of the entropy formula (2.1) shows that the maximum entropy is achieved when the flipping on sites centered at 2 × 2 arrangements of 0-tiles is done independently and with probability 1/2. Consider a generalization of this procedure:

Figure 3 here

The **rule** above which we denote by f_p applied to all neighborhoods in $(2\mathbf{Z})^2$ defines a **probabilistic cellular automaton (pca)**, F_p. It maps (tile) configurations from one square lattice to its dual.

For the golden mean subshift we have $p = 1/2$. By Proposition 2.2. and by the fact that the measure of maximal entropy is unique for this subshift, the marginal of this measure restricted to one of the sublattices is an invariant measure for the pca $F_{1/2}$. The map of course alternates between copies of this measure on configurations on the appropriate sublattices. Hence it is an equilibrium measure in the dynamic sense as well. This naturally suggest that one should investigate the dynamics of F_p, $p \neq 1/2$, and their invariant measures.

The maps F_0 and F_1 are rather simple. The former maps any tile configuration to a all-0-tile configuration in one step and hence preserves only $\delta_{\underline{0}}$. The map F_1^2 clearly preserves both $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$. More generally

Proposition 3.1.: *The pca map F_1 preserves the set of configurations on the diagonal $\rho_e + \rho_o = 1$.*

Proof: By the argument in the beginning of Section 2 the density of mixed 2×2 neighborhoods (not all-0 or all-1) is zero in the configurations on the diagonal. Hence in almost all neighborhoods F_1 flips the assignment and consequently swaps the subsets at (ρ_e, ρ_o) and (ρ_o, ρ_e). ∎

The pca F_ϵ and $F^2_{1-\epsilon}$ for a small ϵ behave essentially like majority voter-models. See [E2] for introduction and basic properties of these in the lattice-dual lattice set up. By $F_{b,p}$ we denote a pca of this type on symbols 0 and 1 that updates a homogeneous 2×2 neighborhood with the other symbol w.p. b, a 3-1-majority loses with probability p and in evenly split neighborhoods the outcome is 1 with probability 1/2.

The equilibrium properties are argued heuristically using results for voter models as follows. The pca F_ϵ clearly preserves a measure close to Bernoulli(ϵ). This rule is exactly the rule $F_{\epsilon,0}$ in neighborhoods with at most one 1 in them. The rules are different in other neighborhoods but these are vanishingly small fraction of neighborhoods in generic configurations at small values of ϵ. F_ϵ disintegrates them immediately and under $F_{\epsilon,0}$ any finite island of 1's of area A is expected to vanish in time $c(\epsilon)A$, $c(\epsilon) \geq 1$ (the rule $F_{0,0}$ is convex confined hence $c(0) = 1$, for details of this see [E2]). The $F_{\epsilon,0}$-rule is critical in ϵ preserving two measures for small ϵ. Hence F_ϵ should preserve a measure close to the lower (low 1-density) invariant measure of $F_{\epsilon,0}$.

The action of the pca $F_{1-\epsilon}$ agrees with $F_{1-\epsilon,0}$ in neighborhoods with at most one 1 and is arbitrarily close (in ϵ) in neighborhoods with exactly four 1's. But they do not agree in other neighborhoods and since neither of these rules preserves a measure close to $\delta_{\underline{0}}$ or $\delta_{\underline{1}}$ the argument above does not hold. However both $F^2_{1-\epsilon}$ and $F^2_{1-\epsilon,0}$ preserve such measures and thereby both have multiple invariant measures (+/−-phases of Ising model).

To confirm this finding and to estimate the critical value p_c above which F_p preserves two invariant measures a series of computer simulations was performed. The configurations were on a lattice of the size 120×120 sites with periodic boundary condition. Initial state was all-0 and the run was continued until the density on the two lattices (i.e. even and odd times) was approximately constant i.e. the system relaxed to a ground state. The runs indicated that F_p is ergodic for $p \leq 3/4$ and nonergodic for $p \geq 13/16 \approx 0.8125$. Note that this interval is almost centered at the estimated critical value 0.791502 in [BET]. The best rigorous lower bound for p_c is far off at 0.575474 ([RS]). With our approach one could perhaps improve it.

Finally we note that the pca formulation of course holds for all of the subshifts in this paper. The neighborhood in the pca rule is of the size of the 0/1-tile. Similar relaxation results are expected.

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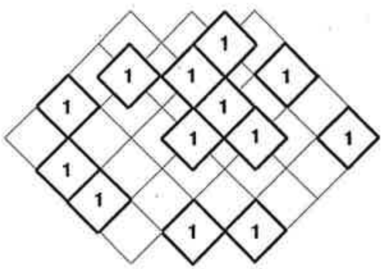

Figure 1. Hard square gas

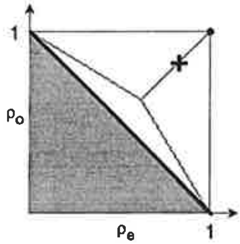

Figure 2. Density diagram

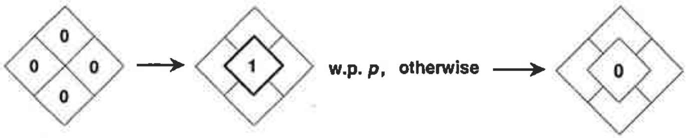

Figure 3. Probabilistic cellular automaton