Dense packing on uniform lattices

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Abstract We study the Hard Core Model on graphs obtained from Archimedean tilings to gain insight into the geometry of the densest packings in a uniform discrete set-up. We establish density bounds, optimal configurations reaching them in all cases, and introduce a probabilistic cellular automaton that generates all legal configurations. Its rule involves a parameter which is naturally characterized as packing pressure. It can have a critical value but from packing point of view just as interesting are the noncritical cases. These phenomena are related to the exponential size of the set of densest packings and more specifically whether these packings are maximally symmetric, random laminated or essentially random packings.

In this project we concentrate on the packing aspects on certain geometric graphs by studying the **Hard Core Model** on them. The model requires each vertex to be occupied by either 0 or 1 but nearest neighbors (hop count one) cannot both be 1's. More specifically we focus on the dense packing regime on planar uniform lattices obtained from **Archimedean tilings** (Figure 1, fine line). There are eleven such lattices but the principles and phenomena observed here extend beyond this class (as indicated e.g. at the bottom of the Table).

Examples the densest possible configurations are illustrated in Figure 1 (solid dots, the indicated configurations are optimal i.e. maximally symmetric). All densest packings divide into three qualitatively different classes (the letter refers to the Table):

- **L : Laminated packings.** Seven of the eleven Archimedean lattices only have a small finite number of optimal packings and these are undeformable.
- **R : 2d-random packings.** Three other lattices allow **local moves** in the optimal packing (akin to domino flip/phason exchange in lozenge tiling) and hence have positive 2d-entropy (residual entropy, positivity implies exponential degeneracy of the ground state).
- **RL: Random laminated packings.** Borderline cases, e.g. the lattice $(3^3, 4^2)$, allowing a **non-local move** and hence having positive 1-dimensional entropy but vanishing 2-dimensional entropy. (Their non-transitivity and long range order is analyzed in [2]. The optimal packing of spheres in 3d Euclidean space is of this type ([1]).)

A probabilistic cellular automaton can be used to generate all packings at a given density level. If all nearest neighbors of a lattice site are 0's, the center of the neighborhood is set to 1 with probability p, otherwise it is forced to 0 (Fig. 2 shows this on the square lattice tiled with diamonds). One PCA iterate is a run through the configurations on all optimal sublattices updating each such sublattice at time. p could be interpreted as **packing pressure**.

The three way division above is reflected in the PCA as an existence of a **dynamic phase transition**. In the laminated case there is a nontrivial critical value p_c above which the Gibbs measure is not unique anymore (densities of 1's on the optimal sublattices differ etc.). Correspondingly when $p \uparrow 1$ the densest packings are supported only by a finite number of optimal sublattices. In the class R there is no critical value and the equilibrium of the PCA is uniform distribution on an exponential number (in the domain size) of densest packings. Sandwiched between these is the RL case in which there also seems to be a dynamic phase transition although a weaker one. In the laminated case the criticality seems to be due an embedded **voter rule** ([3]) in the PCA. Figure 4 indicates the voter behavior of a high pressure rule on honeycomb, triangular and square lattices together with square lattice with Moore neighborhood (left to right, probability of update 1 from the square of the local rule as a function of the number of 1's in the neighborhood).

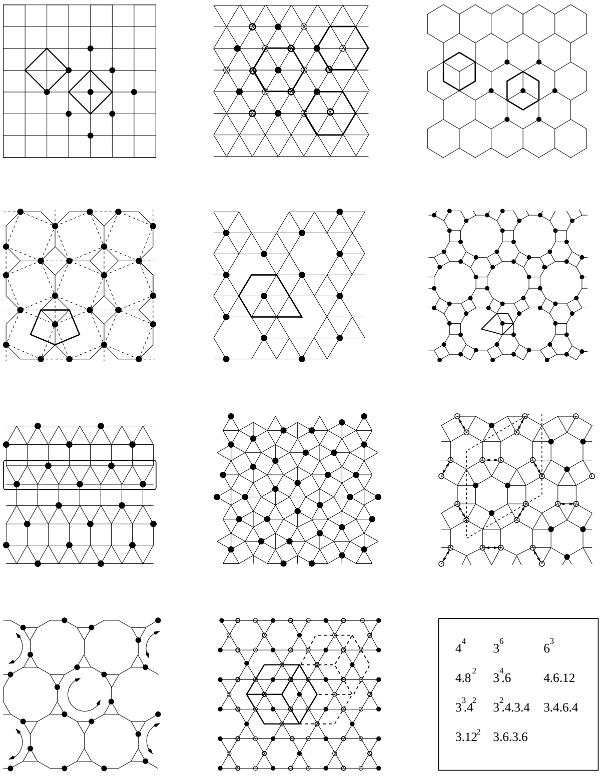
Table indicates the approximate critical probabilities and corresponding densities as well as bounds for the residual entropies in the non-critical cases. The numerical data has been established through simulations on toral lattices of sizes at least 200×200 sites. As an example Figure 3 shows the relaxation of the sublattice densities in one 5000 step run in the subcritical regime ($p = .859$) on the honeycomb lattice.

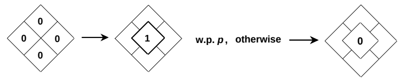
References

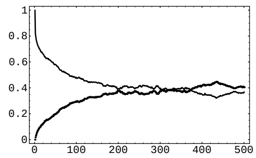
[1] Conway, J.H., Goodman-Strauss, C., Sloane, N.J.A.: Recent progress in sphere packing, *Current Developments in Mathematics*, editors B. Mazur, W. Schmid, S. T. Yau, D. Jerison, I. Singer and D. Stroock, Cambridge, pp. 37-76, 1999.

[2] Eloranta, K.: A note on certain rigid subshifts, *Ergodic Theory of* $\mathbf{Z}^d$*-Actions*, London Mat. Soc. Lect. Notes **228**, Cambridge Univ. Press, pp. 307-317, 1996.

[3] Liggett, T.: *Interacting Particle Systems*, Springer, 1985.







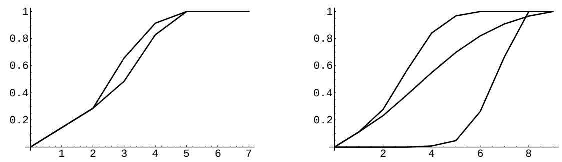


Figure 1, left and **2, 3, 4** right, top to bottom.

Graph	Optimal subgraphs, multiplicity, tiles	Pack. type	ρ	Critical pressures & densities, residual entropies
(4^4) $(\mathbf{Z}^2)$	$\sqrt{2}\ \mathbf{Z}^2$, 2, diamond	L	1/2	$p_c \approx 0.79,\ \rho(p_c) \approx 0.36$
(6^3) $(\mathbf{H})$	$\sqrt{3}\ \mathbf{T}$, 2, hexagon	L	1/2	$p_c \approx 0.87,\ \rho(p_c) \approx 0.4$
(3^6) $(\mathbf{T})$	$\sqrt{3}\ \mathbf{T}$, 3, hexagon	L	1/3	$p_c \approx 0.90,\ \rho(p_c) \approx 0.26$
(4.8^2)	$\simeq (3^2.4.3.4)$, 2, 5-gon	L	1/2	$p_c \approx 0.90,\ \rho(p_c) \approx 0.4$
$(4.6.12)$	$\simeq (3^4.6)$, 2, 5-gon	L	1/2	$p_c \approx 0.91,\ \rho(p_c) \approx 0.42$
$(3^2.4.3.4)$	$\simeq \mathbf{T}$, 3, 6-gon	L	1/3	$p_c \approx 0.99, \rho(p_c) \approx 0.3$
$(3^4.6)$	$\sqrt{3}\ (3^4.6)$, 3, 5-gon	L	1/3	$p_c \approx 0.97,\ \rho(p_c) \approx 0.29$
$(3^3.4^2)$	3 $\mathbf{Z}$ stack, ∞	RL	1/3	$h^{(1)} = \frac{1}{2}\log 2$
$(3.4.6.4)$	$\simeq (3^4.6),\ \infty$, 5-gon	R	1/3	$h^{(2)} \geq \frac{3}{16}\log 2$
$(3.6.3.6)$ $(\mathbf{K})$	$\sqrt{3}\ (3.6.3.6),\ \infty$, rhombus	R	1/3	$h^{(2)} \approx 0.323$
(3.12^2)	$\simeq (3^4.6),\ \infty$, 5-gon	R	1/3	$h^{(2)} \geq \frac{1}{18}\log 2$
$\mathbf{Z}^2$M	2 $\mathbf{Z}$ stack, ∞, square	RL	1/4	$p_c \approx 0.98?,\ h^{(1)} = \frac{1}{2}\log 2$
$[4.8^2]$ $(\mathbf{UJ})$	$\mathbf{Z}^2$, 1, sq., $\sqrt{2}\ \mathbf{Z}^2$, 2, dia.	L	1/2	increasing critical curve
Quilt $(\mathbf{Q})$	$\sqrt{2}\ \mathbf{Z}^2$, 3, diamond	L	1/3	increasing critical curve

Table. Packing properties of the Archimedean and other graphs. The first three lines correspond to the regular graphs, the first 11 to the Archimedean ones and last three are non-Archimedean. All except $\mathbf{Z}^2$M are planar.

- Optimal subgraphs carry the most symmetric densest packings. $\simeq$ means topological equivalence.
- Multiplicity: number of densest packings. Finite number means that all optimal ones are isometric to the most symmetric one, infinite that there are random perturbations.
- Tile: n-gon is not regular, whereas diamond, square and hexagon are. Rhombus forms from two equilateral triangles glued together at base.
- $\rho, h^{(d)}$: density of the optimal packings in the full graph, d-dimensional residual entropy.