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The kink of cellular automaton Rule 18 performs a random walk

Kari Eloranta, Helsinki University of Technology
Esa Nummelin, University of Helsinki

We give an exact characterization of the movement of a single kink in the elementary cellular automaton Rule 18. It is a random walk with independent increments as well as independent delay times. Its statistical parameters are computed to confirm the earlier simulation results by Grassberger [G].

Keywords: Cellular automaton, kink, random walk

0. Introduction

It is quite common in cellular automata that several invariant configurations or phases can be identified. In one dimension the boundaries between these are called kinks or dislocations. In some cases they move in a regular fashion like signals carrying information whereas in some cases their motion is highly erratic reflecting the randomness in the initial configuration. The latter situation has been studied empirically by Grassberger in [**G**] as a model for deterministic diffusion. The 'canonical' case for chaotic kink motion seems to arise in the context of the elementary Rule 18. Knowing this phenomenon would clarify the asymptotic behavior of the system as indicated by Lind [**L**]. Moreover it is likely that by utilizing block transformation equivalences many other one-dimensional cellular automata could then be analyzed analogously to Rule 18. In this note we rigorize the idea of a single kink in Rule 18 performing a random walk and compute its statistical parameters. This confirms the earlier simulation based estimates.

1. Set-up and result

Let $\{0,1\}$ be the set of symbols and $E = \{0,1\}^{\mathbf{Z}}$ be the set of configurations. A *one-dimensional elementary cellular automaton* is a dynamical system on E defined by a blockmap on three neighbouring symbols which commutes with the shift on E. The blockmap of *Rule 18* is simply $001 \mapsto 1$, $100 \mapsto 1$ while other triples map to zero. Here we follow the standard numbering of elementary cellular automata (see e.g. Wolfram [**W**]). Later the Rule 6 on binary doubles is also considered. In order to avoid possible confusion between these rules we call them then 18/256 and 6/16 (there are 16 and 256 elementary rules on binary doubles and triples respectively).

The image of a configuration $\eta = \{\eta(x), x \in \mathbf{Z}\}$ under the Rule 18 is denoted by $\tau\eta$. A partial configuration is denoted by $\eta[a,b] = (\eta(a), \ldots, \eta(b)),\ a \le b,\ a, b \in \mathbf{Z}$.

A configuration η contains a *kink* if $\eta[a,b] = (1,0,\ldots,0,1)$ for $b-a$ odd. The middle of the kink $(a+b)/2$ belongs to $\mathbf{Z}$+1/2.

Here is a simple illustration of the action of Rule 18 on a piece of configuration with a single kink in it:

$$
(1)\qquad
\begin{array}{ll}
\eta & \cdot\ \cdot\ \cdot\ 0\ 0\ 1\ 0\ 1\_1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ \cdot\ \cdot\ \cdot \\
\tau\eta & \cdot\ \cdot\ \cdot\ \cdot\ 1\ 0\ 0\ 0\ 0\ 0\ 0\_0\ 0\ 0\ 0\ 0\ 0\ 1\ \cdot\ \cdot\ \cdot\ \cdot \\
\tau^2\eta & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 1\ 0\ 0\ 0\ 0\ 0\_0\ 0\ 0\ 0\ 0\ 1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \\
\tau^7\eta & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 1\_1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

The middle of the kink has been indicated with an underbar. Note that the middle point first jumps $R - L = 4 - 1 = 3$ steps to the right where $R = 4 =$ the number of 1's to the right of the kink until two zeros and $L = 1 =$ the same number to the left. After this jump the middle stays put for a time $= R + L + 2 = 7$ after which it again jumps.

Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}_0$ the set of non-negative integers. Define a subset of E by

$$F = \{\eta \mid \text{for some } a,\quad \eta(a - 2j + 1) = \eta(a + 2j) = 0,\ \forall j \in \mathbf{N}\}.$$

Then any $\eta \in F$ contains at most one kink. The set F is invariant under the Rule 18.

Let α be the Bernoulli(1/2) distribution on each of the the unspecified coordinates of F. It is easy to see that the subset in F of those configurations that have a kink is of full measure.

Suppose $\{t_i,\ i \in \mathbf{N}_0\}$ are i.i.d. positive random variables. Then $T_i = t_0 + \cdots + t_{i-1},\ T_0 = 0$ is a renewal process on $\mathbf{N}_0$. Let $I(n) = i$ for $T_i \leq n < T_{i+1}$ be the counting process. Let X_0 be a random variable on $\mathbf{Z} + 1/2$ and $\{X_i\}_{i\geq 1}$ an i.i.d. sequence of $\mathbf{Z}$-valued random variables that are independent of X_0. If $\{(X_i, t_i),\ i \in \mathbf{N}_0\}$ are independent pairs (but not necessarily within a pair) then $S_n = X_0 + \cdots + X_{I(n)}$ defines a $\mathbf{Z}+1/2$-valued *random walk with i.i.d. delay times.*

Our result can now be stated. We use the notation $X \stackrel{d}{=} Y$ when random variables X and Y have the same distribution.

Theorem: *Suppose that $\eta \in F$ with a kink is distributed according to α. If S_n denotes the midpoint of the kink in $\tau^n \eta$ then it is a random walk with i.i.d. delay times. In particular the i^{th} jump $X_i \stackrel{d}{=} R - L$ and the holding time $t_i \stackrel{d}{=} R + L + 2$ where R and L are geometrically distributed with parameter $1/2$. The random walk has zero drift and squared variation asymptotically equal to n.*

Remark: By [**L**] Bernoulli(1/2) is the only non-trivial invariant product measure for the Rule 18 on the configurations with every other entry zero. Hence if α is Bernoulli(p_1) distributed with $p_1 \neq 0, 1/2, 1$ on F, the movement of the kink is a non-stationary stochastic process (a random walk in a temporally inhomogeneous medium).

2. The proof

We first simplify the action of the rule on F to its essence. By adding a zero to the kink in $\eta \in F$ we obtain a configuration with at least every other entry zero. These (even or odd indexed zeros) are then removed. On the remaining configuration the rule is now 6/16 on binary doubles. This transformation is analogous to the linearization of Jen [**J**]. Graphically:

$$\begin{array}{cccc} 0\ \ 0 & 0\ \ 1 & 1\ \ 0 & 1\ \ 1 \\ 0 & 1 & 1 & 0 \end{array} \qquad \downarrow \qquad (\nwarrow , \nearrow)$$

The arrow points to the direction of time.

A simple but important observation is that this ruletable has a spatial three-way symmetry i.e. it is identical rule when time is changed to run to either of the directions indicated by the arrows in parenthesis. Or equivalently the rule is *permutive* i.e. fixing the value of any of the cells in the triplet defines a permutation.

The Rule 6/16 is formulated as follows. Let $E_{1/2} = \{0,1\}^{\mathbf{Z}+1/2}$ and $\tilde{E} = E \cup E_{1/2}$, where as before $E = \{0,1\}^{\mathbf{Z}}$. Then the cellular automaton map $\tilde{\tau}$ acts as $\tilde{\tau}\eta(x) = 1$ if $\eta(x-1/2) \neq \eta(x+1/2)$ and 0 otherwise. Hence $\tilde{\tau}(E) = E_{1/2}$ and $\tilde{\tau}(E_{1/2}) = E$. We distribute the initial configuration η on E according to Bernoulli(1/2). It follows that for each even n $\tau^n\eta$ has also Bernoulli(1/2) distribution on E whereas for odd n $\tau^n\eta$ has Bernoulli(1/2) distribution on $E_{1/2}$. Now any $\eta[a,b] = (1,0,\ldots,0,1) \in \tilde{E}$ can be designated to be a kink with middle point at $(a+b)/2$. If $b-a \geq 2$, it's successor is $\eta[a+1/2, b-1/2]$. If $b-a=1$ then the kink jumps and its successor is $\eta[c,d]$ where $c = \max\{\, x \leq a-1/2 |\ \tilde{\tau}\eta(x) = 1\}$ and $d = \min\{\, x \geq b+1/2 |\ \tilde{\tau}\eta(x) = 1\}$.

We illustrate how the particular kink movement described in (1) happens under the transformed rule (again the underbar within the 1–block denotes the middle of the kink):

$$
\text{(2)} \qquad
\begin{array}{ll}
\eta & \cdot\ \cdot\ \cdot\ 0\ 1\ 1\_1\ 1\ 1\ 1\ 1\ 0\ \cdot\ \cdot\ \cdot \\
\tilde{\tau}\eta & \quad\cdot\ \cdot\ \cdot\ 1\ 0\ 0\ 0\_0\ 0\ 0\ 1\ \cdot\ \cdot\ \cdot \\
\tilde{\tau}^2\eta & \cdot\ \cdot\ \cdot\ \cdot\ 1\ 0\ 0\ \underline{0}\ 0\ 0\ 1\ \cdot\ \cdot\ \cdot \\
 & \\
\tilde{\tau}^7\eta & \quad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 1\_1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

Now the kink first moves $(R-L)/2 = (4-1)/2 = 3/2$ steps to the right where $R =$ the number of 1's to the right of the kink until first zero and $L =$ the same number on the left hand side. Like in the rule 18/256 the holding time is $R + L + 2 = 7$.

From the construction it is now clear that the dynamical systems (F, τ) and $(\tilde{E}, \tilde{\tau})$ when started from configurations with one kink are isomorphic. Hence in particular the movement of the kink is identical up to scaling. We shall again use the notation X_i and t_i for the i^{th} jump and holding time of the kink.

The following is the core of the argument.

Lemma: *Let us consider the kink movement in the Rule 6/16 i.e. the system* $(\tilde{E}, \tilde{\tau})$ *starting from a Bernoulli(1/2) distributed* $\eta \in E$*. Suppose that at time* n *we have a* *kink of the form* $(1,1)$ *at* S_n *in* $\tilde{\tau}^n\eta$*. Then the next jump is* $X_{I(n)+1} \stackrel{d}{=} (R-L)/2$ *and the next holding time is* $t_{I(n)+1} \stackrel{d}{=} R+L+2$ *where* $R, L \sim$ *Geom*$(1/2)$ *are independent of each other and independent of the past of the walk.*

Proof: Suppose that $S_n = x$. It is clear that the history of the kink is confined to the backward cone with vertices at (x,n) and $(x \pm (n+1)/2, 0)$ (see the Figure). Or equivalently the past σ–field $\mathcal{F}_n$ of the kink is completely determined by the configurations in the backward cone at (x,n). Moreover $\eta[x-(n+1)/2, x+(n+1)/2]$ and its complement are independent. At step $n \mapsto n+1$ the kink jumps (expands from a $(1,1)$ kink into a wider one). Its right endpoint moves $R+1/2$ steps to the right where R is the number of ones to the right of the kink before the first zero (see the Figure in which $R=3$ and $L=5$). Given $\mathcal{F}_n$, by permutivity the value at $\eta(x+(n+3)/2)$ determines the value of every one of the cells $\tilde{\tau}^i\eta(x+(n-i+3)/2)$, $i=0,1,\ldots,n$. Since $\eta(x+(n+3)/2)$ is independent of $\mathcal{F}_n$ so is $\tilde{\tau}^n\eta(x+3/2)$. This argument iterated implies $R \sim$ Geom$(1/2)$ and its independence of $\mathcal{F}_n$. A symmetric argument yields the distribution of L, the number of ones to the left. The delay time is the height of the new triangle of zeros surrounded by ones which equals to $R+L+2$. ∎

Figure. Movement of the kink

Proof of the Theorem: By the isomorphy of 6/16 and 18/256 on the special configurations it suffices to just consider the system $(\tilde{E}, \tilde{\tau})$. The Lemma yields the i.i.d. increments and delay times. For Rule 18/256 the temporal increment is identical whereas the spatial increment is double (see also illustrations (1) and (2). Obviously $\mathbf{E}(X_1) = 0$ hence the expected drift is

$$\mathbf{E}(S_n - S_0) = \mathbf{E}\left(\sum_{i=1}^{I(n)} X_i\right) = \mathbf{E}(X_1)\mathbf{E}(I(n)) = 0$$

by Wald's identity ($I(n)$ is optional). The expected squared increment and expected delay time are both readily computable from R and L and equal to 4. Since

$\{(X_i^2, t_i), i \in \mathbf{N}_0\}$ are mutually i.i.d. by the Renewal Theorem we get that

$$\frac{1}{n}\mathbf{E}\left(\sum_{i=1}^{I(n)} X_i^2\right) \longrightarrow \frac{\mathbf{E}\left(X_1^2\right)}{\mathbf{E}(t_0)} = 1$$

as $n \to \infty$. ∎

3. Conclusion

In the case of several kinks new phenomena appear. Neighbouring kinks annihilate each other and it is known that from an initial configuration with finite support at most one kink survives after a finite time ([**J**]). However the mechanism for the joint motion of even two kinks seems complicated due to dependency. It needs to be understood well in order to confirm Lind's conjectures and fully understand Rule 18.

References

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