10/1993

Note on certain rigid subshifts

Kari Eloranta*
Helsinki University of Technology

eloranta@janus.hut.fi

Abstract

We study the subshifts of finite type on $\{0,1\}^{\mathbf{Z}^2}$ having the property in every 3×3 square there is a fixed number of ones. In particular we investigate shift-deformations of allowed configurations and show the topological non-transitivity of the shift-action. The last property is also shown to hold for all two-dimensional subshifts of the same generalized type.

Keywords: subshift of finite type, $\mathbf{Z}^2$-action, zero entropy, topological transitivity, tiling
AMS Classification: 52C20, 54H20

Introduction

The space of **configurations** $X = \{0,1\}^{\mathbf{Z}^2}$ together with the product topology is a compact metric space. The following non-empty subsets of X are of interest to us.

Definition: *For $k = 1, 2, 3$ or 4* **the rule** *k is satisfied if in a 3×3 square there are exactly k ones. The set of configurations which satisfy rule k everywhere is denoted by X_k.*

Remark: Properties of rules 5,6,7 and 8 obviously follow from the properties of the defined rules by considering zeros instead of ones.

---

* Research partially supported by the Academy of Finland and The Finnish Cultural Foundation

The 3×3 square in which we impose the consistency requirement according to one of the rules is called a **window** and the piece of configuration in it a **scenery.**

It is natural to equip the configuration space with the horizontal and vertical coordinate shifts

$$\sigma_h(x)_{(i,j)} = x_{(i+1,j)} \quad \text{and} \quad \sigma_v(x)_{(i,j)} = x_{(i,j+1)}, \qquad x \in X.$$

These actions obviously commute. Since X_k are closed under the action of the shifts we obtain **subshifts of finite type** $(X_k, \sigma_h, \sigma_v)$.

Surprisingly little is known about the structure of these spaces and the action on them. In this note we alleviate this situation a little bit. The original motivation (rule 4) was provided by K. Schmidt in [**S**].

1. Spaces

All the defined spaces are non-empty. The allowed or legal configurations are related to each other in ways worth investigating.

There are two different types of periodic configurations: (horizontally or vertically) periodic and doubly-periodic. The first is only invariant under ${\sigma_h}^p {\sigma_v}^q$ for some minimal $(p,q) \neq (0,0)$ (and its multiples), the second under ${\sigma_h}^{p_1} {\sigma_v}^{q_1}$ and ${\sigma_h}^{p_2} {\sigma_v}^{q_2}$ where $(p_i,q_i) \neq (0,0)$ and (p_i,q_i) are not rationally related.

In constructing new configurations from given ones it is useful to realize the following **legal moves** for deforming configurations (and sceneries).

Lemma 1.1.: *Given an allowed configuration in X_k any vertical or horizontal shift of a column or a row of period three results in an allowed configuration.*

The proof is immediate since these shifts leave the number of ones seen in any window unchanged.

Proposition 1.2.: *The configurations in X_1 are either periodic or doubly-periodic. In $X_k,\ k = 2,3,4$ there are also aperiodic configurations.*

Proof: Let $k = 1$. Consider in a legal configuration an arbitrary one say at (x,y) and its nearest neighboring one to the right at $(x+3,y')$ (this is uniquely determined). If they have the same ordinates test $(x+3,y')$ and the one to its right for the same property an so on until a pair with different ordinates is found. If none is found check the left neighbors the same way. If no such pair is found we have a bi-infinite periodic row $\underline{001}$. But this implies that the rows at distance one and two are $\underline{0}$ and the rows at distance three are $\underline{001}$. Hence the global configuration is horizontally periodic or doubly-periodic. If a vertically unaligned pair was found it forces a bi-infinite vertical column of width eight and of period three. Extending this we see that the configuration is vertically periodic but not doubly-periodic.

In the case of rules $2, 3$ and 4 a doubly-periodic configuration is generated for example from the prototiles in shown in Figure 1 a,b and c respectively. All non-trivial rows are period three so a shift by one to the right of any such row gives a horizontally periodic configuration which is not vertically periodic. In these configurations there are still period three columns (through the middle of a deformed tile). Hence a vertical shift by one results in an aperiodic legal configuration. ∎

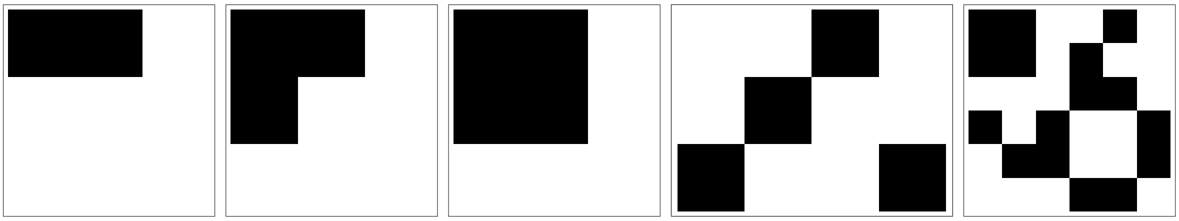

Figure 1 a,b,c,d,e. Prototiles. Black is one, white is zero.

We require prototiles to be rectangles. Doubly-periodic configurations of course have the key property of being immediately globally constructible from the local description of the tile. Periodic but non-doubly periodic configurations cannot be generated from a single tile but they may be generated from a tile and its periodic shift as in the proof above. As the proof indicates some of the aperiodic configurations are also generated from a small number of prototiles.

The following result resolves the possible sizes of prototiles.

Theorem 1.3.: *For X_1 the prototile generating a doubly-periodic configuration is of size 3×3. For X_k, $k = 2$ or 4 it can be of any size $3w \times 3h$ where w, h are natural numbers. For X_3 all prototiles have at least one of the sidelengths a multiple of three.*

Proof: By the structure of configurations in X_1 the first assertion is clear. For $k = 2$ or 4 consider a legal segment of height 3 and width l which is period three horizontally. Let its i^{th} row contain n_i ones. Then necessarily $\sum_{i=1}^{3} n_i/(3l) = k/9$ which is the density of ones in any strip of width three. But this implies that l is divisible by 3. For a $l_1 \times l_2$ prototile apply this argument along any column and row of width 3 to obtain the second part of the statement. By using legal column and row shifts as in the proof of Proposition 1.2. it is easy to show that prototiles with any w, h can be constructed. If $k = 3$ the density argument applied to a $l_1 \times l_2$ prototile implies that one of the sidelengths is necessarily divisible by three. ∎

Remark: In X_3 there are indeed configurations generated from prototiles with one of the sidelengths not a multiple of three, e.g. the one shown in Figure 1d.

The Theorem naturally leads to the question whether any allowed configuration is a result of a sequence of legal moves starting from a single doubly-periodic configuration. In other words is X_k a **shift-connected** set? By the proof of Proposition 1.2. this is clearly the case for X_1. The question is of some subtlety since seemingly the only systematic way of constructing a global configuration is by legal moves from a doubly-periodic configuration.

Turns out that X_4 is not shift-connected. After solving (by computer) all the legal 6×6 prototiles a few "exotic" ones like the one shown in Figure 1e. were found. In the global configuration it generates there are only rows and columns of period six hence it cannot be shift-deformed to a period three configuration.

The question remains open for X_2 and X_3. As a hint to the affirmative it was found that in every 6×6 tile for X_2 and in every 6×4, 6×5, 6×6 and 6×7 tile for X_3 there are period three rows or columns.

2. Action

We now consider the basic properties of the (σ_h, σ_v)-action on the configuration spaces.

If in a window we see n_1 ones in the leftmost, n_2 in the center and n_3 in the rightmost column we call it **type** (n_1, n_2, n_3).

The following observation is of some consequence.

Lemma 2.1.: *Let x be a configuration in X_k, any k. Given a scenery in x with column distribution (n_1, n_2, n_3) consider the bi-infinite horizontal strip of width three containing it. Every scenery in this strip has column distribution (n_1, n_2, n_3) or it's cyclic permutation. Analogously for the scenery's row-distribution.*

Proof: By shifting the window to the right by one the scenery loses n_1 ones hence the same number has to enter from the right. ∎

It is easy to see that each X_k, $k > 1$, contains configurations with k adjacent (horizontal or vertical) ones somewhere.

Proposition 2.2.: *For rule 2 no configuration with two adjacent horizontal ones can contain two adjacent vertical ones. For rules 3 and 4 no configuration with three adjacent horizontal ones can contain three adjacent vertical ones.*

Proof: The argument is identical for all three cases so let us consider $k = 2$. Let H be the horizontal strip of height three containing a scenery with two vertical adjacent ones and let L be the vertical strip of width three containing a scenery with two horizontal adjacent ones. By applying Lemma 2.1. we see that in $H \cap V$ there are both two adjacent horizontal and vertical ones. Hence the rule 2 is violated. ∎

For each $X_k,\ k > 1$, let $X_k^{(h)}$ and $X_k^{(v)}$ be the set of configurations containing somewhere at least $k \wedge 3$ horizontally or vertically adjacent ones. Let $X_k^{(c)}$ be the complement of their union. In view of Proposition 2.2. these three sets partition the configuration space. Subsequently we drop the index k once it has declared which rule is considered.

The **orbit** of an element in X_k is the set $O(x) = \left\{\sigma_h^i \sigma_v^j(x) |\ (i,j) \in \mathbf{Z}^2\right\}$. Denote its closure by $\overline{O(x)}$. A continuous transformation on a metric space is called **topologically transitive** if there is a element in the space whose orbit under the transformation is dense ([**W**]). Note that the shift-action on any X_k is continuous in the inherited topology so it make sense to ask whether this minimal mixing prevails in our set-up.

Recall that X can be metrized by many equivalent ways e.g. by

$$d(x,x) = 0 \quad \text{and} \quad d(x,y) = 2^{-\min\left\{\max\{|i|,|j|\}|\ x_{(i,j)} \neq y_{(i,j)}\right\}} \quad \text{for} \quad x \neq y.$$

Hence for denseness it suffices to consider matching on arbitrarily large squares centered at origin.

Theorem 2.3.: *None of the systems* $(X_k, \sigma_h, \sigma_v)$, $k = 1, 2, 3$ *or* 4 *is topologically transitive.* X_1 *has two maximal transitive components; the vertically and horizontally periodic configurations. For* $X_k,\ k > 1$, *the following exclusions hold:*
$X^{(h)} \cap \overline{O(x)} = \emptyset\ \forall x \in X^{(v)}$ *and* $X^{(v)} \cap \overline{O(x)} = \emptyset\ \forall x \in X^{(h)}$ *together with*
$X^{(h)} \cap \overline{O(x)} = \emptyset$ *and* $X^{(v)} \cap \overline{O(x)} = \emptyset\ \forall x \in X^{(c)}$.
For $k > 2$ *we also have the weaker exclusion:* $X^{(c)} \not\subset \overline{O(x)}\ \forall x \in X^{(h)} \cup X^{(v)}$.

Proof: In view of Proposition 1.2. we know that every vertically (or doubly-) periodic configuration is uniquely determined by specifying how much (± 1 or 0) every third column is shifted with respect to its neighbor three to the left. This set of vertical off-sets $\{-1, 0, 1\}^{\mathbf{Z}}$ is in a one-to-one correspondence with the set of vertically (and doubly-) periodic configurations upto a horizontal shift to match the non-zero columns. Call a legal column of width three with all ones in the left column a slab. The desired x is built simply by placing first slabs with different off-sets next to each other, then next to them pairs of slabs with all different pairs of off-sets and so on. Hereby we exhaust all finite blocks of off-sets and consequently x must have a dense orbit in vertically periodic configurations.

For $k > 1$ it is an immediate consequence of the definition of $X^{(c)}$ that the orbits of its elements are bounded away from elements in both $X^{(h)}$ and $X^{(v)}$. Moreover the exclusion property of Proposition 2.2. clearly implies that no element in $X^{(h)}$ $\left(X^{(v)}\right)$ can have its orbit arbitrarily close to any element in $X^{(v)}$ $\left(X^{(h)}\text{ respectively}\right)$. For $k > 2$ this can be further refined as follows. Note that for each element x in $X^{(h)}$ every scenery must by Lemma 2.1. be of the type $(1, 1, 1)$ for rule 3 and $(2, 1, 1)$ or its cyclic permutation for rule 4. But then $O(x)$ is bounded away from those configurations in $X^{(c)}$ in which every scenery is of type $(2, 1, 0)$ or $(2, 2, 0)$ respectively (for example the doubly periodic configurations generated by the prototiles in Figure 1b and 1c). ∎

Remarks: 1. Also in the $k > 1$ case at least two of the transitive components are simply related: the one in the complement of $X^{(h)}$ is of course just a 90 degree rotation of the one outside $X^{(v)}$. Whether these or the other components further split by a more subtle exclusion/conservation law remains open.

2. The results indicate that configurations are indeed quite rigid. Another sign of this is that the systems (any k) are of zero topological entropy. This follows from the observation that a diagonal strip of width four determines the entire configuration. But the number of different such diagonals of length N is proportional to 2^{cN}. Hence the number of different configurations in a $N \times N$ square is not proportional to $2^{c'N^2}$ and the entropy must be zero.

3. Extension

The methods used for Theorem 2.3. suitably refined actually enable one to prove the basic non-transitivity result in greater generality. We present it here and give a streamlined proof indicating the modifications.

Suppose that we have $s < \infty$ symbols $S = \{0, 1, 2, \ldots, s-1\}$ and consider the subsets of $S^{\mathbf{Z}^2}$ defined by the **rule** $(k_0, k_1, \ldots, k_{s-1})$ which requires that in every $p \times p$ square there are exactly k_i copies of the symbol i. Obviously $k_i \geq 1$ and $\sum k_i = p^2$. Call the subspace $X_{(k_0,k_1,\ldots,k_{s-1})}$.

Theorem 3.1.: *The shift-action on any of the spaces* $X_{(k_0,k_1,\ldots,k_{s-1})}$ *is topologically non-transitive.*

Proof: If $k_i = 1$ for some symbol i the argument is as in Theorem 2.3. Suppose that $k_i \geq 2\ \forall i$. Given a symbol i define $X^{(h)}(i)$ for it essentially as before: it contains all configurations in which there is somewhere at least $k_i \wedge p$ i's in a horizontal p-block. The exclusion argument of Proposition 2.2. works for this symbol if $2\,(k_i \wedge p) - 1 > k_i$. This is equivalent to $1 < k_i < 2p - 1$ holding. Hence if k_i satisfies these inequalities the sets $X^{(h)}(i)$ and $X^{(v)}(i)$ are disjoint (and contain the orbits starting from them). As before elements in the complement of their union cannot have dense orbits in either set and the non-transitivity follows.

The remaining (and novel) case is the one where $k_i < 2p-1$ fails for all symbols. Note that then we consider (p, s)-pairs with $s \leq p^2/(2p-1)$. Given a symbol i the configurations in $X^{(h)}(i)$ have in every scenery the symbol i-distributions of the form $(n_1, n_2, \ldots, n_p)$, $\forall n_l \geq 1$ (or its cyclic permutation). On the other hand $X^{(c)}(i)$ clearly contains e.g. doubly periodic configurations where the i-distribution is of the form $(n'_1, n'_2, \ldots, 0)$ in every scenery. This is just because $k_i \geq 2p - 1\ \forall i$ so in particular any symbol can form an `L` -shaped area like the symbols 0 does in Figure 1c. Hence elements of $X^{(h)}(i)$ cannot have dense orbits in $X^{(c)}(i)$ and the converse holds again by definition. ∎

The type-preservation in shifting the window seems to produce strong enough argument only in two dimensions. The natural generalization of the definition for $X^{(h)}$ for $\mathbf{Z}^d$-actions, $d \geq 3$ is not useful for this reason and it is unclear how to argue the higher dimensional case.

Acknowledgement

The author would like to thank Mats Nordahl for stimulating discussions on some of these problems.

References

[**S**] Schmidt, K.: *Algebraic ideas in ergodic theory*, CBMS Reg. Conf. Ser. in Math. **76**, American Mathematical Society, 1990.

[**W**] Walters, P.: *An introduction to ergodic theory*, Springer-Verlag, 1982.